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On the exponential Diophantine equation $73^{i} + 127^{j} = k^2$

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Abstract
For ages, academics have been fascinated with Diophantine equations. These problems, which are named after the Greek mathematician Diophantus, entail solving algebraic equations with integers. These equations have been used extensively in many areas of mathematics in the contemporary era, such as applied algebra, coordinate geometry, and trigonometry. The Pythagorean Theorem, which asserts that the square of the hypotenuse's length in a right-angled triangle equals the sum of the squares of the lengths of the other two sides, is among the most well-known instances of a Diophantine equation. Finding integer answers to this theorem, which can be written as a Diophantine equation, is a well-known mathematical issue. Finding answers to Diophantine equations can be extremely difficult, despite their significance. There are no universal techniques for resolving every Diophantine equation, in contrast to many other kinds of mathematical problems. Finding solutions is a difficult and time-consuming task because each equation must be tackled individually. In this paper we look for the solutions of the exponential Diophantine equation $73^{i} + 127^{j} = k^2$, where i, j, k whole numbers are. The objective of the study is to ascertain whether this specific equation has any non-negative integer solution and to investigate any patterns or connections found in the solution set. The researchers are able to discover some fascinating facts about the nature of exponential Diophantine equations by looking into certain instances of the problem and applying sophisticated mathematical methods. The study clarifies the difficulties in resolving such issues and emphasizes the significance of further research in this field.

Keywords: Gross Prime number, Whole numbers, Diophantine equation, Solution.

2020 Mathematics Subject Classification: 11D61, 11D72.

1. Introduction

Since Diophantine equations pose particular difficulties in determining integer solutions, they have been a major area of study in number theory for decades. Innovative findings and applications in a variety of mathematical ideas and real-world issues have resulted from the study of Diophantine equations. In several fields, including coding theory, cryptography, and optimization issues, linear Diophantine equations of the form $ax + by = c$, where a, b, and c are integers, have been well explored [1]. Non-linear Diophantine equations, on the other hand, present more difficult problems and need advanced methods to resolve, since they contain higher degree terms or many variables. Understanding the nature of real numbers and the structure of mathematical systems depends on the presence of irrational numbers, which are demonstrated by these equations [2–3].

Mathematicians have discovered profound links between various areas of mathematics and created new methods for problem-solving by investigating the solutions to Diophantine equations. In

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summary, Diophantine equations have substantial applications in a variety of domains in addition to their importance in number theory. Mathematicians are still motivated to push the limits of mathematical knowledge and investigate the intricate relationships between various mathematical notions via their study. Acu [4] discussed the Diophantine equation $2^x + 5^y = z^2$. Sroysang [5-6] studied the Diophantine equations $8^x + 19^y = z^2$, and $31^x + 32^y = z^2$. Rabago [7] studied an open problem of Diophantine equation that was given by B. Sroysang. Sroysang [8] analyzed the Diophantine equation $8^x + 13^y = z^2$. Kumar et al. [9] examined the non-linear Diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$ for their non-negative integer solutions. Kumar et al. [10] also studied the two non-linear Diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$. Aggarwal et al. [11] examined the Diophantine equation $223^x + 241^y = z^2$ and proved that this equation is not solvable in the set of whole numbers. The Diophantine equation $181^x + 199^y = z^2$ was examined by Aggarwal et al. [12] for the existence of its solution. Kumar et al. [13] proved that the exponential Diophantine equation $601^p + 619^q = r^2$ has no solution in the set of non-negative integers. Mishra et al. [14] studied the Diophantine equation $211^\alpha + 229^\beta = \gamma^2$ and showed that there do not exist non-negative integers α, β, and γ that satisfy this Diophantine equation. Goel et al. [15] considered the exponential Diophantine equation ${M_5}^p + {M_7}^q = r^2$ and prove that this equation is not solvable in the set of whole numbers. Kumar et al. [16] examined the exponential Diophantine equation $(2^{2m+1} - 1) + (6^{r+1} + 1)^n = \omega^2$. Bhatnagar and Aggarwal [17] proved that the exponential Diophantine equation $421^p + 439^q = r^2$ is not solvable in the set of non-negative integers.

Kumar et al. [18] studied the exponential Diophantine equation $(7^{2m}) + (6r + 1)^n = z^2$ and determined that this Diophantine equation is not solvable in the set of whole numbers. Aggarwal and Sharma [19] examined the non-linear Diophantine equation $379^x + 397^y = z^2$ for its non-negative integer solution. Aggarwal [20] determined the non-negative integer solution of the Diophantine equation$193^x + 211^y = z^2$. The non-linear Diophantine equation $313^x + 331^y = z^2$ was examined by Aggarwal et al. [21]. Aggarwal et al. [22] also studied the non-linear Diophantine equation $331^x + 349^y = z^2$ and discovered that this equation has no solution in the set of non-negative integers. Aggarwal and Kumar [23] examined the exponential Diophantine equation $(13^{2m}) + (6r + 1)^n = z^2$ and showed that this Diophantine equation is not solvable in the set of whole numbers. Aggarwal and Kumar [24] proved that there exist no p, q, r in the set of non-negative integers that satisfy the exponential Diophantine equation $439^p + 457^q = r^2$. Aggarwal and Kumar [25] examined the exponential Diophantine equation $(7^{2m}) + (6^{r+1} + 1)^n = \omega^2$ for its non-negative integer solutions. Aggarwal and Kumar [26] solved the exponential Diophantine equation $(2^{2m+1} - 1) + (6r + 1)^n = z^2$ completely. The exponential Diophantine equation $(2^{2m+1} - 1) + (13)^n = z^2$ was examined by Aggarwal [27]. Aggarwal and Kumar [28] determined the non-negative integer solution of the exponential Diophantine equation ${M_3}^p + {M_5}^q = r^2$. The exponential Diophantine equation $(19^{2m}) + (6^{\gamma+1} + 1)^n = \rho^2$ was studied by Aggarwal and Kumar [29].

Aggarwal and Kumar [30-32] determined the non-negative integer solutions of the Diophantine equations$(19^{2m}) + (12\gamma + 1)^n = \rho^2$, $(19^{2m}) + (6\gamma + 1)^n = \rho^2$, and $[19]^{2m} + [2^{2r+1} - 1] = \rho^2$. Aggarwal et al. [33] provided the solution to the Diophantine equation $143^x + 45^y = z^2$ in the set of non-negative integers. Gupta et al. [34] obtained the non-negative integer solution of the non-linear exponential Diophantine equation $x^\alpha + (1 + my)^\beta = z^2$. Solution of the non-linear exponential Diophantine equation $(x^a + 1)^m + (y^b + 1)^n = z^2$ was given by Gupta et al. [35]. Aggarwal and Upadhyaya [36] studied on the Diophantine equation $8^\alpha + 67^\beta = \gamma^2$. Solution of the Diophantine equation $22^x + 40^y = z^2$ was also provided by Aggarwal and Upadhyaya [37]. Aggarwal et al. [38] examined the Diophantine equation $143^x + 85^y = z^2$, where x, y, and z were non-negative integers. Solution of the Diophantine equation $143^x + 485^y = z^2$ was given by Aggarwal et al. [39]. Aggarwal et al. [40] discussed the problem of the solution of the non-linear (exponential) Diophantine equation$\beta^x + (\beta + 18)^y = z^2$. Aggarwal [41] solved completely the Diophantine equation $323^x + 85^y = z^2$ in his study. Solution of the Diophantine equation $783^x + 85^y = z^2$ was given by Aggarwal and Upadhyaya [42]. Aggarwal et al. [43] solved the exponential Diophantine equation $10^x + 400^y = z^2$ in the set of non-negative integers. The Diophantine equation $8^{i} + 71^{j} = k^2$ was examined by Aggarwal et al. [44]. Aggarwal and Shahida [45] provided the non-negative integer solution of the Diophantine equation $10^x + 40^y = z^2$. Solution of the exponential Diophantine equation $n^x + 43^y = z^2$, where $n \equiv 2 (mod\ 129)$ and $n + 1$ is not a perfect square, was given by Aggarwal and Shahida [46].

Exploring the complexities of the exponential Diophantine equation $73^i + 127^j = k^2$ is the main goal of this study. The structure of the equation entails the interaction of whole integers i, j, k as we attempt to solve the problem of its answers. We aim to investigate whether and what kind of solutions exist for this specific problem.

2. Preliminaries

We state two important lemmas here that will be essential in helping us successfully tackle the main issue that we are looking into in this study.

Lemma 1. *The exponential Diophantine equation* $73^i + 1 = k^2$, *where* i, k *are whole numbers, has no solution in whole numbers.*

Proof: Since 73 is an odd prime therefore 73^i is an odd number for all whole numbers i.

$\Rightarrow 73^i + 1 = k^2$ is an even number for all whole numbers i.

$\Rightarrow k$ is an even number.

$\Rightarrow k^2 \equiv 0(mod 3)$ or $k^2 \equiv 1(mod 3)$ (1)

Now, $73 \equiv 1(mod 3)$

$\Rightarrow 73^i \equiv 1(mod 3)$, for any whole number i

$\Rightarrow 73^i + 1 \equiv 2(mod 3)$, for all whole numbers i

$\Rightarrow k^2 \equiv 2(mod 3)$. (2)

Equation (2) contradicts equation (1). Hence the exponential Diophantine equation $73^i + 1 = k^2$, where i, k are whole numbers, has no solution in whole numbers.

Lemma 2. *The exponential Diophantine equation* $127^j + 1 = k^2$, *where* j, k *are whole numbers, has no solution in whole number.*

Proof: Since 127 is an odd prime, so 127^j is an odd number for all whole numbers j.

$\Rightarrow 127^j + 1 = k^2$ is an even number for all whole numbers j.

$\Rightarrow k$ is an even number.

$\Rightarrow k^2 \equiv 0(mod 3)$ or $k^2 \equiv 1(mod 3)$. (3)

Now, $127 \equiv 1(mod 3)$

$\Rightarrow 127^j \equiv 1(mod 3)$, for all whole numbers j

$\Rightarrow 127^j + 1 \equiv 2(mod 3)$, for all whole numbers j

$\Rightarrow k^2 \equiv 2(mod 3)$. (4)

Equation (4) contradicts equation (3). Hence the exponential Diophantine equation $127^j + 1 = k^2$, where j, k are whole numbers, has no solution in the set of whole numbers.

3. Main Theorem: The exponential Diophantine equation $73^i + 127^j = k^2$, where i, j, k are whole numbers, has no solution in the set of whole numbers.

Proof: There are four cases:

Case: 1 If $i = 0$ then the exponential Diophantine equation $73^i + 127^j = k^2$ becomes $1 + 127^j = k^2$, which has no whole number solution by Lemma 2.

Case: 2 If $j = 0$ then the exponential Diophantine equation $73^i + 127^j = k^2$ becomes $73^i + 1 = k^2$, which has no whole number solution by lemma 1.

Case: 3 If $i,j>0$, then $73^{i}, 127^{j}$ are odd numbers.

$\Rightarrow 73^{i}+127^{j}=k^2$ is an even number

$\Rightarrow k$ is an even number

$\Rightarrow k^2 \equiv 0(mod3)\ or\ k^2 \equiv 1(mod3)$ (5)

Now, $73 \equiv 1(mod3)$

$\Rightarrow 73^{i} \equiv 1(mod3)\ and\ 127 \equiv 1(mod3)$

$\Rightarrow 73^{i} \equiv 1(mod3)\ and\ 127^{j} \equiv 1(mod3)$

$\Rightarrow 73^{i}+127^{j} \equiv 2(mod3)$

$\Rightarrow k^2 \equiv 2(mod3)$ (6)

Equation (6) contradicts equation (5). Hence the exponential Diophantine equation $73^{i}+127^{j}=k^2$ has no whole number solution in this case.

Case: 4 If $i,j=0$, then $73^{i}+127^{j}=1+1=2=k^2$, which is impossible because k is a whole number. It shows that the exponential Diophantine equation $73^{i}+127^{j}=k^2$ has no whole number solution in this case.

Hence the exponential Diophantine equation $73^{i}+127^{j}=k^2$, where i,j,k are whole numbers, has no solution in the set of whole numbers.

4. Conclusion

The authors of this particular study thoroughly examined the complexities of solving the exponential Diophantine problem $73^{i}+127^{j}=k^2$ in their research. In order to clarify the basic characteristics of such equations, they carefully investigated the particular case in which i,j, and k are whole numbers throughout their work. In spite of their prolonged efforts, the authors finally came to a noteworthy conclusion that the exponential Diophantine equation $73^{i}+127^{j}=k^2$ has no solution in the set of whole numbers.

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