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On the solution of the exponential Diophantine equation $(43)^{2x} + (6r + 1)^y = z^2$, where r, x, y, and z are whole numbers

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Abstract
Nowadays mathematicians are highly interested in discovering new and advanced methods for determining the solutions of Diophantine equations. Diophantine equations are those equations which have more unknowns than the number of equations. Diophantine equations appear in astronomy, knot theory, cryptography, abstract algebra, coordinate geometry and trigonometry. Congruence theory plays an important role in finding the solution of some special type Diophantine equations. The absence of any generalized method, which can handle each Diophantine equation, is a very challenging problem for researchers. In the present paper the authors discuss the existence of the solution of the exponential Diophantine equation $(43)^{2x} + (6r + 1)^y = z^2$, where r, x, y, and z are whole numbers. Results of the present paper show that the exponential Diophantine equation $(43)^{2x} + (6r + 1)^y = z^2$, where r, x, y, and z are whole numbers, has no solution in whole numbers.

Keywords: Diophantine equation, Solution, Whole numbers, Odd numbers, Even numbers.

2020 Mathematics Subject Classification: 11D61, 11D72.

1. Introduction

Diophantine equations have various applications in solutions to the problems of science and mathematics such as balancing the chemical reactions, the Die Hard problem, finding the Pythagorean triples, and the Brahmagupta problem [1-2], etc. There are various types of Diophantine equations such as linear, non-linear, quadratic, cubic, quintic, homogeneous, non-homogeneous, and exponential [3-5]. There exists a large list of methods to solve Diophantine equations [6-7]. But there is no universal method that solves all types of Diophantine equations. So researchers have solved various types of Diophantine equations by different methods. Sroysang [8-10] has done a good work on Diophantine equations and he solved the Diophantine equations $8^x + 19^y = z^2$, $31^x + 32^y = z^2$, and $8^x + 13^y = z^2$. Rabago [11] solved completely a problem of Diophantine equations that was suggested by Sroysang. Aggarwal and Shahida [12-13] together worked on Diophantine equations $10^x + 40^y = z^2$, and $n^x + 43^y = z^2$, where $n \equiv 2(mod\ 129)$ and $n + 1$ is not a perfect square. They solved completely these two equations in the set of non-negative integers. Aggarwal and Upadhyaya [14-16] examined three Diophantine equations namely, $8^\alpha + 67^\beta = \gamma^2$, $22^x + 40^y = z^2$, and $783^x + 85^y = z^2$. Aggarwal et al. [17] discussed the solution of the Diophantine equation $8^i + 71^j = k^2$. Aggarwal et al. [18] solved completely the non-linear exponential Diophantine equation $\beta^x + (\beta + 18)^y = z^2$. Aggarwal [19-20] determined the solution of the Diophantine equations $(2^{2m+1} - 1) + (13)^n = z^2$ and $193^x + 211^y = z^2$. The main aim of this paper is to examine the exponential

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Diophantine equation $(43)^{2x} + (6r + 1)^{y} = z^{2}$, where $r, x, y,$ and z are whole numbers, for whole number solutions.

2. Preliminaries

We develop in this section some preliminary results which will form the backbone of our work in the succeeding main section of this paper.

Lemma 1. The exponential Diophantine equation $(43)^{2x} + 1 = z^{2}$, where x and z are whole numbers, has no solution in whole numbers.

Proof: Since $(43)^{2x}$ is an odd number for all whole numbers x,
$\Rightarrow (43)^{2x} + 1 = z^{2}$ is an even number for all whole numbers x.
$\Rightarrow z$ is an even number.
$\Rightarrow z^{2} \equiv 0(mod3)$ or $z^{2} \equiv 1(mod3)$. (1)
Now, $43 \equiv 1(mod3)$
$\Rightarrow (43)^{2x} \equiv 1(mod3)$ for all whole numbers x.
$\Rightarrow (43)^{2x} + 1 \equiv 2(mod3)$ for all whole numbers x.
$\Rightarrow z^{2} \equiv 2(mod3)$. (2)
As equation (2) contradicts equation (1), hence the exponential Diophantine equation $(43)^{2x} + 1 = z^{2}$, where x and z are the whole numbers, has no solution in whole numbers.

Lemma: 2 The exponential Diophantine equation $1 + (6r + 1)^{y} = z^{2}$, where $r, y,$ and z are whole numbers, has no solution in whole numbers.

Proof: Since $(6r + 1)$ is an odd number for all whole numbers r so $(6r + 1)^{y}$ is an odd number for all whole numbers r and y.
$\Rightarrow 1 + (6r + 1)^{y} = z^{2}$ is an even number for all whole numbers r and y.
$\Rightarrow z$ is an even number.
$\Rightarrow z^{2} \equiv 0(mod3)$ or $z^{2} \equiv 1(mod3)$. (3)
Now $(6r + 1) \equiv 1(mod3)$ for all whole number r.
$\Rightarrow (6r + 1)^{y} \equiv 1(mod3)$ for all whole numbers r and y.
$\Rightarrow 1 + (6r + 1)^{y} \equiv 2(mod3)$ for all whole numbers r and y.
$\Rightarrow z^{2} \equiv 2(mod3)$. (4)
Again we find that equation (4) contradicts equation (3), therefore, the exponential Diophantine equation $1 + (6r + 1)^{y} = z^{2}$, where $r, y,$ and z are whole numbers, has no solution in whole numbers.

3. Main Theorem: The exponential Diophantine equation $(43)^{2x} + (6r + 1)^{y} = z^{2}$, where $r, x, y,$ and z are whole numbers, has no solution in whole numbers.

Proof: Four distinct cases are possible, which we discuss separately as below:

Case: 1 If $x = 0$, then the exponential Diophantine equation $(43)^{2x} + (6r + 1)^{y} = z^{2}$ becomes $1 + (6r + 1)^{y} = z^{2}$, which has no whole number solution by Lemma 2.

Case: 2 If $y = 0$, then the exponential Diophantine equation $(43)^{2x} + (6r + 1)^{y} = z^{2}$ becomes $(43)^{2x} + 1 = z^{2}$, which has no whole number solution by Lemma 1.

Case: 3 If x and y are positive integers, then $(43)^{2x}$ and $(6r + 1)^{y}$ are both odd numbers.
$\Rightarrow (43)^{2x} + (6r + 1)^{y} = z^{2}$ is an even number.
$\Rightarrow z$ is an even number.
$\Rightarrow z^{2} \equiv 0(mod3)\ or\ z^{2} \equiv 1(mod3)$. (5)

Now, $43 \equiv 1(mod3)$
$\Rightarrow (43)^{2x} \equiv 1(mod3)$ and $(6r + 1) \equiv 1(mod3)$
$\Rightarrow (43)^{2x} \equiv 1(mod3)$ and $(6r + 1)^{y} \equiv 1(mod3)$

$\Rightarrow (43)^{2x} + (6r+1)^y \equiv 2(mod 3)$
$\Rightarrow z^2 \equiv 2(mod 3).$ (6)

Obviously, equation (6) contradicts equation (5). Hence the exponential Diophantine equation $(43)^{2x} + (6r+1)^y = z^2$, where x and y are positive integers, and r, z are whole numbers, has no solution in whole numbers.

Case: 4 If $x = y = 0$, then $(43)^{2x} + (6r+1)^y = 1 + 1 = 2 = z^2$, which is impossible because z is a whole number. Hence exponential Diophantine equation $(43)^{2x} + (6r+1)^y = z^2$, where $x = y = 0$ and r, z are whole numbers, has no solution in the set of whole numbers.

4. Conclusion

The authors showed that the exponential Diophantine equation $\mathbf{(43)^{2x} + (6r+1)^y = z^2}$, where $\boldsymbol{r}, \boldsymbol{x}, \boldsymbol{y}$, and $\boldsymbol{z}$ are whole numbers has no solution in the set of whole numbers.

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